13th International Conference on Applications of Statistics and Probability in Civil Engineering, ICASP13
Seoul, South Korea, May 26-30, 2019

High-Order Moment Method for Structural Reliability Analysis Including Random Variables with Unknown Distributions

Zhao-Hui Lu
Professor, Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing, China

Chao-Huang Cai
Graduate Student, School of Civil Engineering, Central South University, Changsha, China

Chun-Qing Li
Professor, School of Engineering, RMIT University, Melbourne, VIC 3000, Australia

Yan-Gang Zhao
Professor, Department of Architecture, Kanagawa University, Yokohama, Japan (Corresponding author: Zhao@kanagawa-u.ac.jp)

ABSTRACT: The high-order moment methods, being very simple, are widely used for structural reliability analysis. The basic procedure of moment methods includes two steps: firstly, the first few moments of the performance function were determined using the new point estimate method; secondly, the corresponding failure probability can be obtained from the moment-based reliability index. In the new point estimate method, the basic random variables are assumed having known probability distributions to realize the Rosenblatt transformation and its inverse transformation. However, in practical applications, the probability distributions of some random variables maybe unknown, and the probabilistic characteristics of these variables maybe expressed using only statistical moments. This paper aims to investigate the high-order moment methods including random variables with unknown probability distribution based on the fourth-moment transformation technique. Several examples are examined under different conditions to demonstrate the accuracy and efficiency of the present method. Since only the first few moments of the performance functions are used, and it can be conducted even when the probability distributions of the random variables are unknown, structural reliability analysis should become simpler and more convenient using the present method.

1. INTRODUCTION

The search for an effective structural reliability method has led to the development of various reliability approximation techniques, such as the first-order reliability method (FORM) (Hasofer and Lind 1974; Rackwitz and Fiessler 1978; Shinozuka 1983), the second-order reliability method (SORM) (Der Kiureghian et al. 1987; Der Kiureghian and De Stefano 1991; Cai and Elishakoff 1994; Zhao and Ono 1999), importance sampling Monte Carlo simulation (Melchers 1990; Fu 1994), first-order third-moment reliability method (FOTM) (Tichy 1994; Zhao and Ang 2012), response surface approach (Faravelli 1989; Rajashekhar and Ellingwood 1993; Liu and Moses 1994; Allaix and Carbone 2011), high-order moment methods (Zhao and Ono 2000; Zhao and Ono 2001; Zhao and Lu 2007; Lu et al. 2017) and so on.

For the high-order moment methods (Zhao and Ono 2001), one can first estimate the probability moments of the performance function using the new point estimate method (Zhao and Ono 2000), and the corresponding failure probability can then be obtained from the moment-based reliability index. The high-order moment methods, being very simple, have no shortcomings with respect to design points, and require neither iteration nor the computation of derivatives. A literatures review shows that the moment methods are widely used for structural reliability analysis (e.g., Lu et al. 2010; Low 2013; Kong et al. 2014; Wang et al. 2014; Ma et al. 2015).

However, for the new point estimate method, the basic random variables are still assumed having known probability density function (PDF) or cumulative distribution function (CDF) to realize the normal transformation (*x*-*u* transformation) and its inverse transformation (*u*-*x* transformation) with the aid of Rosenblatt transformation. In practical applications, the probability distributions of some random variables maybe unknown, and the probabilistic characteristics of these variables maybe expressed using only statistical moments. In such circumstances, the Rosenblatt transformation cannot be applied, and a strict evaluation of the probability of failure is not possible. For this reason, the objective of this paper is to investigate the high-order moment methods including random variables with unknown probability distribution.

In this study, the fourth-moment transformation technique, in which the *u*-*x* and *x*-*u* transformations directly use the first four moments of the random variables instead of their probability distributions, is combined with the new point estimates method to compute the first four moments of the performance function. The efficiency and accuracy of the present method for structural reliability assessment including random variables with unknown CDF/PDFs is demonstrated through several numerical examples.

2. ESTIMATING MOMENTS OF PERFORMANCE FUNCTIONS INCLUDING RANDOM VARIABLES WITH UNKNOWN PROBABILITY DISTRIBUTIONS

2.1. Point Estimates for Function of Single Variable

For a performance function of only one random variable, $Z=G(\mathbf{X})$, the first four statistical moments of the performance function are theoretically expressed as the following integrals:

$$\mu_G = E[G(\mathbf{X})] = \int G(x) f(x) dx \tag{1a}$$

$$\sigma_G^2 = E\{[G(\mathbf{X}) - \mu_G]^2\} = \int [G(x) - \mu_G]^2 f(x) dx \tag{1b}$$

$$\sigma_G^3 \alpha_{3G} = E\{[G(\mathbf{X}) - \mu_G]^3\} = \int [G(x) - \mu_G]^3 f(x) dx \tag{1c}$$

$$\sigma_G^4 \alpha_{4G} = E\{[G(\mathbf{X}) - \mu_G]^4\} = \int [G(x) - \mu_G]^4 f(x) dx \tag{1d}$$

where $G(\mathbf{X})$ is the performance function, which is a single random variable as well as a function of basic random variable X; μ_G, σ_G, α_{3G} and α_{4G} are the first four moments, i.e., mean, standard deviation, skewness and kurtosis of the performance function; and $f(x)$ is the PDF of basic random variable $\mathbf{X}$.

For some simple functions, the first few moments can be directly obtained from the definitions in Eq. (1). However, in practice, because $G(\mathbf{X})$ is generally a complicated and implicit function, the computation of Eq. (1) by using direct integration is inconvenient.

On the other hand, the new point estimates method, in which any random variables were required to be transformed into standard normal variables with the aid of Rosenblatt transformation, has been developed (Zhao and Ono 2000) for evaluating probability moments of an arbitrary performance function of random variable. The new point estimates, being very simple, have no computation of derivatives, and can remove the weaknesses of other point estimates method. However, in practical applications, the probability distributions of some random variables which are necessary when using Rosenblatt transformation maybe unknown, and the probabilistic characteristics of these variables maybe expressed using only statistical moments. For this reason, the fourth-moment transformation technique (Fleshimen 1978; Zhao and Lu 2007), in which the *u-x* and *x-u* transformations directly use the first four moments of the random variables instead of their probability distributions, is applied.

Without loss of generality, a random variable X can be standardized as follows:

$$X_s = \frac{X - \mu_X}{\sigma_X} \tag{2}$$

where μ_X and σ_X are the mean value and standard deviation of X, respectively. Based on the fourth-moment transformation (Zhao and Lu 2007), X_s can be expressed as a third-order polynomial of the standard normal random variable U, i.e.,

$$X_s = S_u(U) = -l_1 + k_1 U + l_1 U^2 + k_2 U^3 \tag{3}$$

where the coefficients of l_1, k_1 and k_2 can be given by (Zhao and Lu 2007):

$$l_1 = \frac{\alpha_{3X}}{6(1+6l_2)} \tag{4a}$$

$$k_1 = \frac{1-3l_2}{1+l_1^2-l_2^2} \tag{4b}$$

$$k_2 = \frac{l_2}{1+l_1^2+12l_2^2} \tag{4c}$$

in which

$$l_2 = \frac{1}{36}\left(\sqrt{6\alpha_{4X} - 8\alpha_{3X}^2 - 14} - 2\right) \tag{4d}$$

where α_{3X} and α_{4X} are the skewness and kurtosis of X, respectively. From Eq. (3), the *x-u* transformation is readily obtained as

$$U = -\frac{\sqrt[3]{2}p}{\sqrt[3]{-q+\Delta}} + \frac{\sqrt[3]{-q+\Delta}}{\sqrt[3]{2}} - \frac{l_1}{3k_2} \tag{5}$$

where

$$\Delta = \sqrt{q^2 + 4p^3} \tag{6a}$$

$$p = \frac{3k_1k_2 - l_1^2}{9k_2^2} \tag{6b}$$

$$q = \frac{2l_1^3 - 9k_1k_2l_1 + 27k_2^2(-l_1 - X_s)}{27k_2^3} \tag{6c}$$

Using the polynomial normal transformation shown above, the first four moments of $G(\mathbf{X})$ shown in Eq. (1) can be rewritten as

$$\mu_G = \int G[\sigma_X S_u(u) + \mu_X]\phi(u)du \tag{7a}$$

$$\sigma_G^2 = \int \{G[\sigma_X S_u(u) + \mu_X] - \mu_G\}^2 \phi(u)du \tag{7b}$$

$$\sigma_G^3\alpha_{4G} = \int \{G[\sigma_X S_u(u) + \mu_X] - \mu_G\}^3 \phi(u)du \tag{7c}$$

$$\sigma_G^4\alpha_{4G} = \int \{G[\sigma_X S_u(u) + \mu_X] - \mu_G\}^4 \phi(u)du \tag{7d}$$

where $\phi(u)$ is the PDF of the standard normal random variable.

Let $L(U) = G[\sigma_X S_u(U)+\mu_X]$, using the Gauss-Hermite integration with m points, the first four central moments of $G(\mathbf{X})$ can be calculated by

$$\mu_G = \sum_{j=1}^{m} P_j \cdot L(u_j) \tag{8a}$$

$$\sigma_G^2 = \sum_{j=1}^{m} P_j \cdot [L(u_j) - \mu_G]^2 \tag{8b}$$

$$\sigma_G^3\alpha_{3G} = \sum_{j=1}^{m} P_j \cdot [L(u_j) - \mu_G]^3 \tag{8c}$$

$$\sigma_G^4\alpha_{3G} = \sum_{j=1}^{m} P_j \cdot [L(u_j) - \mu_G]^4 \tag{8d}$$

where $u_1, u_2, \ldots, u_m$ are the estimating points, and $P_1, P_2, \ldots, P_m$ are the corresponding weight. For a seven point estimate in standard normal space, they are given by

$$u_{3-} = -3.7504397;\ P_{3-} = 5.48269 \times 10^{-4} \tag{9a}$$

$$u_{2-} = -2.3667594;\ P_{2-} = 3.07571 \times 10^{-2} \tag{9b}$$

$$u_{1-} = -1.1544054;\ P_{1-} = 0.2401233 \tag{9c}$$

$$u_0 = 0;\ P_0 = 16/35 \tag{9d}$$

$$u_{1+} = 1.1544054;\ P_{1+} = 0.2401233 \tag{9e}$$

$$u_{2+} = 2.3667594; \quad P_{2+} = 3.07571 \times 10^{-2} \tag{9f}$$

$$u_{3+} = 3.7504397; \quad P_{3+} = 5.48269 \times 10^{-4} \tag{9g}$$

2.2. Point Estimates for Function of Multiple Variables

The procedure for a single random variable described above can be extended to a performance function of many variables. For a performance function $Z=G(\mathbf{X})$, where $\mathbf{X}=(X_1, X_2, \ldots, X_n)$, the estimating points would be in the m^n hyperquadrants of the space defined by the n random variables, where m is the number of estimating points. However, the computation becomes excessive when n is large. In this section, an approximation approach is applied, in which the n-dimensional performance function is approximated by the summation of a series of, at most, D-dimensional functions ($D < n$).

Here, the one-dimension reduction ($D = 1$) are introduced. Using the third-order polynomial transformation, a performance function $Z = G(\mathbf{X})$ can be rewritten as:

$$L(\mathbf{U}) = G\left[\sigma_{X_1} S_u(U_1) + \mu_{X_1}, \cdots, \sigma_{X_n} S_u(U_n) + \mu_{X_n}\right] \tag{10}$$

in which $\mathbf{U}$ denotes an independent standard normal random vector; μ_{X_i} and σ_{X_i} are the mean and standard deviation of X_i (i = 1, 2, …, n), respectively; and $S_u()$ is a third-order polynomial of standard normal random variable.

According to the one-dimension reduction method (Zhao and Ono 2000a; Xu and Rahman 2004), $L(\mathbf{U})$ can be approximated as

$$L(\mathbf{U}) \cong L_1 + (n-1)L_0 \tag{11}$$

where

$$L_0 = L(0, \ldots, 0, \ldots, 0) \tag{12a}$$

$$L_1 = \sum_{i=1}^{n} L(0, \ldots, U_i, \ldots, 0) = \sum_{i=1}^{n} L_i(U_i) \tag{12b}$$

Because U_i are independent standard normal random variables and $L_i(U_i)$ is a function only of U_i, $L_i(U_i)$ (i = 1, 2, …, n) are also independent. The first four moments of $L(\mathbf{U})$ can be given as

$$\mu_G = \sum_{i=1}^{n} \mu_i + (n-1)L_0 \tag{13a}$$

$$\sigma_G^2 = \sum_{i=1}^{n} \sigma_i^2 \tag{13b}$$

$$\alpha_{3G}\sigma_G^3 = \sum_{i=1}^{n} \alpha_{3i}\sigma_i^3 \tag{13c}$$

$$\alpha_{4G}\sigma_G^4 = \sum_{i=1}^{n} \alpha_{4i}\sigma_i^4 + 6\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \sigma_i^2\sigma_i^2 \tag{13d}$$

where μ_i, σ_i, α_{3i}, and α_{4i} are the mean, standard deviation, skewness, and kurtosis of $L_i(U_i)$. Since $L_i(U_i)$ is a function of only one standard normal random variable, the first four moments μ_i, σ_i, α_{3i}, and α_{4i} can be point estimated from Eq. (8).

3. THE FOURTH-MOMENT RELIABILITY INDEX

After the first four moments of the performance function have been determined, the failure probability (P_f) can be estimated from

$$P_f = \Phi(-\beta_{4M}) \tag{14}$$

where β_{4M} is the fourth-moment reliability index (Lu et al. 2017), which is given by

$$\beta_{4M} = \frac{\sqrt[3]{2}p}{\sqrt[3]{-q_0 + \Delta_0}} - \frac{\sqrt[3]{-q_0 + \Delta_0}}{\sqrt[3]{2}} + \frac{l_1}{3k_2} \tag{15}$$

where

$$\Delta_0 = \sqrt{q_0^2 + 4p^3} \tag{16a}$$

$$p = \frac{3k_1k_2 - l_1^2}{9k_2^2} \tag{16b}$$

$$q_0 = \frac{2l_1^3 - 9k_1k_2l_1 + 27k_2^2(-l_1 + \beta_{2M})}{27k_2^3} \tag{16c}$$

in which

$$l_1 = \frac{\alpha_{3G}}{6(1+6l_2)} \tag{17a}$$

$$k_1 = \frac{1-3l_2}{1+l_1^2 - l_2^2} \tag{17b}$$

$$k_2 = \frac{l_2}{1 + l_1^2 + 12 l_2^2} \tag{17c}$$

$$l_2 = \frac{1}{36}\left(\sqrt{6\alpha_{4G} - 8\alpha_{3G}^2 - 14} - 2\right) \tag{17d}$$

$$\beta_{2M} = \frac{\mu_G}{\sigma_G} \tag{17e}$$

4. NUMERICAL EXAMPLES

4.1. Example 1

The I beam is widely employed in the practical engineering, such as mechanical engineering, civil engineering, and vehicle engineering to provide maximal strength in one direction (usually the direction bearing weight) with a minimal expense of material. An I beam is subject to a concentrate force P with a distance A away from the fixed end as depicted in Figure 1 (Huang and Du 2006; Huang and Zhang 2013).

The performance function is given by

$$G(\mathbf{X}) = S - \frac{12P\mathrm{a}(L-a)d}{2L[b_f d^3 - (b_f - t_w)(d - 2t_f)^3]} \tag{18}$$

The probability distribution information of all eight random variables is shown in Table 1. The normal, lognormal, Gumbel, and Weibull distributions commonly met in engineering problems are considered.

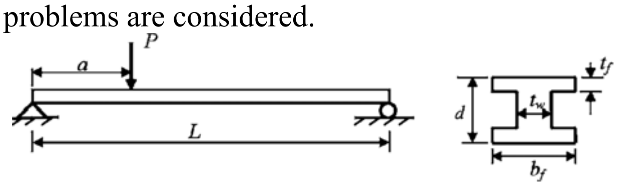

Figure 1: An I beam

Table 1: Random variables and their distribution parameters

Variables	Distribution	Mean	C.O.V	Skewness	Kurtosis
S (MPa)	Lognormal	1.7×10^5	0.028	0.084	3.013
P (N)	Gumbel	5070	0.039	1.1395	5.4
a (mm)	Lognormal	72	0.083	0.251	3.112
L (mm)	Normal	120	0.05	0	3
d (mm)	Normal	2.3	0.018	0	3
b_f (mm)	Normal	2.3	0.018	0	3
t_w (mm)	Normal	0.16	0.130	0	3
t_f (mm)	Normal	0.26	0.080	0	3

Table 2: Comparison of analysis results for Ex.1

Method	P_f	β
MCS(2×10^7samples)	2.83×10^{-3}	2.767
HOMM (Using probability distributions)	3.09×10^{-3}	2.738
The present method (HOMM using statistical moments)	3.09×10^{-3}	2.738

The analysis results are listed in Tables 2. From Table 2, it can be observed that: (1) The results obtained using the present method is almost the same with those obtained using HOMM with full distributions; (2) The results provided by both HOMM are in good agreement with the exact one given by MCS.

4.2. Example 2

Consider an annular disk of inner radius R_i, outer radius R_0, and constant thickness $t \ll R_0$ (plane stress) under the angular velocity ω as shown in Figure 2 (Kang et al. 2016). According to the SAE G-11 standard, the burst margin M_b is defined as a function of material utilization factor α_m, ultimate strength S_u, mass density ρ, geometric and angular velocity variables as

$$M_b = \sqrt{\frac{3\alpha_m S_u (R_0 - R_i)}{\rho[\omega \frac{2\pi}{60}]^2 (R_0{}^3 - R_i{}^3)}} \tag{19}$$

and allowable critical threshold value of the burst margin is 0.37473 yielding the performance function given as

$$G(\mathbf{X}) = 0.37473 - M_b(X_1, X_2, X_3, X_4, X_5, X_6) \tag{20}$$

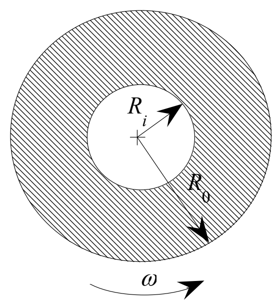

Figure 2: Rotating hooped disk subject to angular velocity

Statistical properties of the random variables in Eq. (20) are listed in Table 3. The results of the probability of failure and corresponding reliability index calculated by using full distribution methods, such as MCS, FORM, together with the corresponding moment-based reliability index obtained by using HOMM with CDF/PDFs of basic random variables, and those obtained by using HOMM with statistical moments of basic random variables, are listed in Table 4. Again, it can be concluded that the results obtained using the present method is in great agreement with those obtained using HOMM with full distributions, and are close to the result of MCS, which is assumed to be exact.

Table 3: Distributions of random variables for Ex.2

Variables	Distribution	Mean	C.O.V	Skewness	Kurtosis
$X_1 = \alpha_m$	Weibull	0.9377	0.049	-0.9225	4.4663
$X_2 = S_u$ (MPa)	Normal	1516.9	0.023	0	3.0
$X_3 = \omega$ (r/min)	Normal	21000	0.0476	0	3.0
$X_4 = \rho$ (g/cm^3)	Uniform	8.027	0.02	0	1.8
$X_5 = R_0$ (cm)	Normal	60.96	0.021	0	3.0
$X_6 = R_i$ (cm)	Normal	20.32	0.0375	0	3.0

Table 4: Comparison of analysis results for Ex. 2

Method	P_f	β
MCS(10^7samples)	1.0×10^{-3}	3.090
HOMM (Using probability distributions)	9.4×10^{-4}	3.107
The present method (HOMM using statistical moments)	9.4×10^{-4}	3.110

5. CONCLUSIONS

(1) In the present paper, the high-order moment method including random variables with unknown probability distribution is proposed. New point estimates method including random variables with unknown CDF/PDFs is developed, eliminating the necessity of using the Rosenblatt transformation in which the probability distributions of all random variables must be known.

(2) Numerical examples results indicate that the present method has high accuracy and can be applied to the performance function including random variables with unknown CDF/PDFs. Since only the first few statistical moments of basic random variables easier obtained from limited experimental data than their PDF/CDFs have been used, the present method are expected to be convenient to be applied to structural reliability analysis. Thus it is an effective alternative to the structural reliability analysis in practical engineering.

(3) Since the fourth-moment transformation and fourth-moment reliability index is only determined by a third-order polynomial of normal random variable, the proposed method may give unsatisfactory results when strong non-normality random variables or performance functions involved in structural reliability analysis. Further study on the applicable ranges and corresponding relative errors of the fourth-moment transformation and fourth-moment reliability index is needed in future.

6. ACKNOWLEDGEMENTS

The study is partially supported by the National Natural Science Foundation of China (Grant Nos.: 51820105014, 51738001). The support is gratefully acknowledged.

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